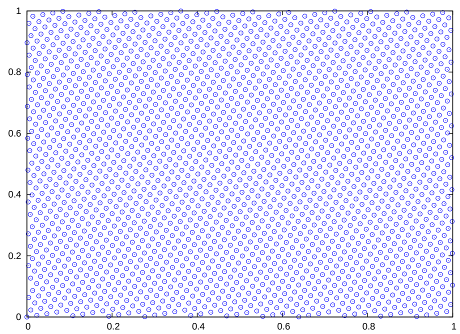

Fig. 1. Point set constructed on [0,1) using rational approximants to γ

1 Introduction

Since the early work of Zaremba [1] in developing good lattice rules using ratios of successive terms of the Fibonacci series, there have been numerous applications to numerical integration, particularly with regards to spherical codes. Ratios of succesive elements of the Fibonacci sequence give rational approximations to the golden ratio γ, whose continued fraction representation is given by:

$$\gamma = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}} \tag{1}$$

or $[0; 1, 1, 1, 1...]$. The golden ratio is frequently referred to as 'the most irrational number', and rational approximations to γ are used in dynamics todescribe winding numbers for stable orbits, in the presence of chaos. There is a distinct analogy between the phase space of dynamics, and point set generation for Monte Carlo integration. In a Poincare map of a chaotic system like the Lorentz attractor, points corresponding to stable orbits appear in regular patterns like quasi-random numbers. Chaotic orbits produce seemingly random intersections with the map, analogous to pseudo random numbers. With this analogy in mind, and exploiting the periodicity of modular arithmetic, one is led to consider the use of rational approximations to γ in the construction of quasi-random point sets on [0,1). The application to two dimensions is obvious, using succesive elements of the Fibonacci sequence:

$$x_1 = (987 * r \bmod 2047)/2047$$

$$x_2 = (610 * r \bmod 2047)/2047$$

$$\text{with } r = 1, ..., 2047 \tag{2}$$

Figure 1 is an example of a point set generated in this manner mod 2047. Using software [2] to calculate a lower bound to discrepancy, this point set has a discrepancy of 7e-4 versus 1.3e-3 for the sobol sequence. While the results are promising, the extension of a good lattice rule to n dimensions is not obvious [3]. The inverse of irrational approximants have the same length continued fraction. However, this still implies that there are ${}^{n}C_2 = (n^2 - n)/2$ ratios of integers to be chosen as 'good' approximants to irrational numbers. For instance, in five dimensions, this implies the optimization of ten rational approximants:

$$\begin{array}{llll} \frac{a}{b} & \frac{a}{c} & \frac{a}{d} & \frac{a}{e} \\ \frac{b}{c} & \frac{b}{d} & \frac{b}{e} & \\ \frac{c}{d} & \frac{c}{e} & & \\ \frac{d}{e} & & & \end{array} \tag{3}$$

By writing the ratios in this manner, it is apparant that all ratios outside the first column may be expressed as products of these entries, for instance:

$$\frac{b}{d} = \frac{b}{c} \cdot \frac{c}{d}$$

$$\frac{a}{d} = \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \tag{4}$$

Therefore, by choosing these $(n-1)$ values as ratios of approximants for irrational numbers, their products should also display similar properties. An obvious seed value for a/b is the approximation to γ used previously eg., 610/987. Proceeding down column one of (3), if b/c is chosen also as an approximation to γ, then this implies:

$$\frac{b}{c} \approx \frac{a}{b} \text{ or } c = \frac{b^2}{a} \approx 1597 \tag{5}$$

The subsequent ratio $\frac{b}{c} = 987/1597$ has a continued fraction expansion of $[0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]$. Defining in a similar manner:

$$\frac{c}{d} \approx \frac{a}{b} \text{ or } d = \frac{b^3}{a^2} \approx 2584 \tag{6}$$

one finds for the ratio $\frac{c}{d} = 1597/2584$ which has a continued fraction expansion one digit longer than b/c[1] . The resulting approximations in the second column

[1] This also points to an interesting identity for Fibonacci series terms above F_{16}; $\frac{b^{(i+1)}}{a^i} = F_{16+i}; i = 1, ..., N$

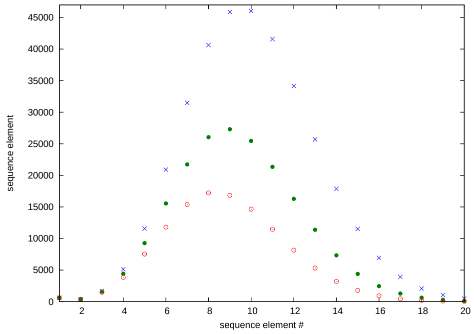

Fig. 2. Integers for lattice generation, 20 dims

also have the desired, long continued fractions (CF), eg.,

$$\gamma^2 \approx \frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = 610/1597 \tag{7}$$

has CF representation $[0; 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. However, the quality of the approximations to irrational numbers decreases with order, which is to be expected, eg.,

$$\gamma^3 \approx \frac{a}{d} = 610/2584 \tag{8}$$

has a CF with period 6. In order to take into account the decreasing quality of approximations to irrational numbers and thus poorer discrepancy of resultant point sets, the column entries apart from the seed may be equated with some function of γ:

$$\frac{f \cdot i}{(n-1)} \frac{a}{b} \tag{9}$$

where as before, a/b is the seed approximant, index i ranges from 2 to $n-1$ and f is a factor determined by minimizing the difference ϵ between rational approximants to irrational numbers. This function would suggest that appart from the seed values, elements of a low-discrepancy point set are given by the function:

$$\text{floor}\left[\frac{b^i (n-1)^{(i-1)}}{i!(af)^{(i-1)}}\right] \tag{10}$$

This function is plotted in figure 2 for various f.

2 Results

$f == 0.99$, 10 dims
`test_seq(10,gen_seq(10,233,144,0.99));` gives 0.019 for discrp. (Sobol==0.018)
$f == 1.3$, 8 dims
`test_seq(8,gen_seq(8,233,144,1.3));` gives 0.016 (Sobol==0.015)

WORK TODO:

(1) optimize f, probably non-convex cost function
(2) higher dims, larger fields
(3) 'scrambled' sequences for variance, use Zech permutations over $GF(2^k)$(?)

References

[1] S. K. Zaremba, *Annali di Matematica Pura ed Applicata* **73**, 293 (1966).

[2] E. Thiemard, *Journal of Complexity* **17**, 850 (2001).

[3] H. Niederreiter, *Random number generation and quasi-Monte Carlo methods*, SIAM, 1992.