

# 1 Analysis for Numerical Verification

Solving for 2 harmonic boundaries 1d fluid motion, under the low reynolds number approximation. The simplest case of constant, symmetric and anti-symmetric boundaries are solved first, and then using Duhammel's principle, time varying boundaries are introduced. This example will serve as a test for a 1D projection of the higher-D numerical solver. The low reynolds number approximate equation (stokes):

$$\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 \phi}{\partial r^2} \quad (1)$$

2 boundaries, 2 harmonic conditions...BC's

$$\begin{aligned} u(-1, t) &= 1 \\ u(1, t) &= 1 \end{aligned} \quad (2)$$

and

$$u(r, 0) = 0; -1 \leq r \leq 1 \quad (3)$$

Take Laplace transform of diffusion equation  $\mathcal{L}(u) \rightarrow U$ :

$$\frac{d^2 U}{dr^2} - \frac{\rho}{\eta} s U = 0 \quad (4)$$

put  $\alpha = \frac{\rho}{\eta}$

$$U = A \cosh(\sqrt{\alpha s} r) + B \sinh(\sqrt{\alpha s} r) \quad (5)$$

BC's

$$U(r = -1) = A \cosh(\sqrt{\alpha s}) - B \sinh(\sqrt{\alpha s}) = 1/s \quad (6)$$

$$U(r = 1) = A \cosh(\sqrt{\alpha s}) + B \sinh(\sqrt{\alpha s}) = 1/s \quad (7)$$

one finds that  $B = 0$  and  $A = 1/s \cosh(\sqrt{\alpha s})$  and hence:

$$U(r, s) = \frac{\cosh(\sqrt{\alpha s}) r}{s \cosh(\sqrt{\alpha s})} \quad (8)$$

Invert using the Hadamard relationship:

$$\mathcal{L}^{-1} \left\{ \frac{P(s)}{Q(s)} \right\} = \sum_{s_k} \frac{P(s_k)}{Q'(s_k)} e^{s_k t}; Q(s = s_k) = 0 \quad (9)$$

One finds:

$$s_k = 0, -\frac{(2n+1)^2 \pi^2}{4\alpha}; n = 0, 1, 2, 3, .. \quad (10)$$

∴

$$u(r, t) = \sum_{s_k} \frac{\cosh(\sqrt{\alpha s} r)}{\cosh(\sqrt{\alpha s}) + \sqrt{\alpha s} \sinh(\sqrt{\alpha s})} e^{s_k t} \quad (11)$$

$$= 1 + \frac{2}{\pi} \sum_{n=0,1,2,\dots} \frac{(-1)^n \cos\left(\frac{[2n+1]\pi r}{2}\right)}{2n+1} \exp\left[-\frac{(2n+1)^2 \pi^2}{4\alpha} t\right] \quad (12)$$

new BC's; let

$$U(r = -1) = A \cosh(\sqrt{\alpha s}) - B \sinh(\sqrt{\alpha s}) = -1/s \quad (13)$$

$$U(r = 1) = A \cosh(\sqrt{\alpha s}) + B \sinh(\sqrt{\alpha s}) = 1/s \quad (14)$$

one finds that  $A = 0$  and  $A = 1/s \sinh(\sqrt{\alpha s})$  and hence:

$$U(r, s) = \frac{\sinh(\sqrt{\alpha s}) r}{s \sinh(\sqrt{\alpha s})} \quad (15)$$

One finds:

$$s_k = 0, -\frac{(n)^2 \pi^2}{\alpha}; n = 0, 1, 2, 3, \dots \quad (16)$$

$\therefore$

$$u(r, t) = \sum_{s_k} \frac{\sinh(\sqrt{\alpha s} r)}{\sinh(\sqrt{\alpha s}) + \sqrt{\alpha s} \cosh(\sqrt{\alpha s})} e^{s_k t} \quad (17)$$

$$= \frac{1}{\pi} \sum_{n=0,1,2,\dots} \frac{(-1)^n \sin(n\pi r)}{n} \exp\left[-\frac{(n)^2 \pi^2}{\alpha} t\right] \quad (18)$$

Introduce harmonic boundaries ( $f(t) = \sin(\omega t)$ ;  $g(t) = \sin(\Omega t + \phi)$ ) s.t:

$$U(r = -1) = F(s) = A \cosh(\sqrt{\alpha s} r) - B \sinh(\sqrt{\alpha s} r) \quad (19)$$

$$U(r = 1) = G(s) = A \cosh(\sqrt{\alpha s} r) + B \sinh(\sqrt{\alpha s} r) \quad (20)$$

one finds

$$A = \frac{F(s) + G(s)}{2 \cosh(\sqrt{\alpha s})} = \frac{H(s)}{\cosh(\sqrt{\alpha s})} \quad (21)$$

$$B = \frac{G(s) - F(s)}{2 \sinh(\sqrt{\alpha s})} = \frac{I(s)}{\sinh(\sqrt{\alpha s})} \quad (22)$$

ie.,

$$U(s, r) = H(s) s \frac{\cosh(\sqrt{\alpha s} r)}{s \cosh(\sqrt{\alpha s})} + I(s) s \frac{\sinh(\sqrt{\alpha s} r)}{s \sinh(\sqrt{\alpha s})} \quad (23)$$

One can use

$$\mathcal{L}[w_t] = sW - w(x, 0) \quad (24)$$

to show that for a general function  $u(x, t)$ :

$$u(x, t) = \mathcal{L}^{-1}\{F(s) \mathcal{L}[w_t]\}$$

$$\begin{aligned}
&= \mathcal{L}^{-1}[F(s)] * [w_t] \\
&= \int_0^t f(\tau) w_\tau(x, t - \tau) d\tau \\
&\int_0^t w(x, t - \tau) f'(\tau) d\tau + f(0)w(x, t)
\end{aligned} \tag{25}$$

Can use solutions from earlier; need to evaluate following integral(s):

$$\begin{aligned}
&\omega \int e^{-\beta(t-\tau)} \cos(\omega\tau) d\tau \\
&= \frac{\omega}{\beta} e^{\beta(\tau-t)} \cos(\omega\tau) \Big|_0^t + \frac{\omega^2}{\beta} \int_0^t e^{\beta(\tau-t)} \sin(\omega\tau) d\tau \\
&= \frac{\omega\beta}{\omega^2 + \beta^2} \left[ e^{\beta(\tau-t)} \cos(\omega\tau) \Big|_0^t + \frac{\omega}{\beta} e^{\beta(\tau-t)} \sin(\omega\tau) \Big|_0^t \right] \\
&= \frac{\omega\beta}{\omega^2 + \beta^2} \left[ \cos(\omega t) - e^{-\beta t} + \frac{\omega}{\beta} \sin(\omega t) \right]
\end{aligned}$$

$$\begin{aligned}
&\Omega \int e^{-\beta(t-\tau)} \cos(\Omega\tau + \phi) d\tau \\
&= \frac{\Omega}{\beta} e^{\beta(\tau-t)} \cos(\Omega\tau + \phi) \Big|_0^t + \frac{\Omega^2}{\beta} \int_0^t e^{\beta(\tau-t)} \sin(\Omega\tau + \phi) d\tau \\
&= \frac{\Omega\beta}{\Omega^2 + \beta^2} \left[ e^{\beta(\tau-t)} \cos(\Omega\tau + \phi) \Big|_0^t + \frac{\Omega}{\beta} e^{\beta(\tau-t)} \sin(\Omega\tau + \phi) \Big|_0^t \right] \\
&= \frac{\Omega\beta}{\Omega^2 + \beta^2} \left[ \cos(\Omega t + \phi) - e^{-\beta t} \cos(\phi) \right. \\
&\quad \left. + \frac{\Omega}{\beta} \{ \sin(\Omega t + \phi) - e^{-\beta t} \sin(\phi) \} \right]
\end{aligned}$$

$$\mathcal{L}^{-1}H(s) = \frac{f(t) + g(t)}{2} \tag{26}$$

$$\mathcal{L}^{-1}I(s) = \frac{g(t) - f(t)}{2} \tag{27}$$

Let  $u(r, t) = A + B$  and for simplicity,  $\omega = \Omega$ ;  $A$  :

$$\frac{\sin(\omega t) + \sin(\omega t + \phi)}{2} + \frac{1}{\pi} \sum_{n=0,1,2,\dots} \frac{(-1)^n \cos(\frac{[2n+1]\pi r}{2})}{2n+1} \times$$

$$\frac{\omega\beta}{\omega^2 + \beta^2} \left[ \cos(\omega t) + \cos(\omega t + \phi) - e^{-\beta t} (1 + \cos(\phi)) \right]$$

$$\begin{aligned}
& + \frac{\omega}{\beta} \left\{ \sin(\omega t) + \sin(\omega t + \phi) - e^{-\beta t} \sin(\phi) \right\} \Big] \\
& + \sin(\phi) + \frac{2 \sin(\phi)}{\pi} \sum_{n=0,1,2,\dots} \frac{(-1)^n \cos\left(\frac{[2n+1]\pi r}{2}\right)}{2n+1} \exp \left[ -\frac{(2n+1)^2 \pi^2}{4\alpha} t \right] \quad (28)
\end{aligned}$$

where  $\beta = \frac{(2n+1)^2 \pi^2}{4\alpha}$ . Similarly, for  $B$ , one finds:

$$\begin{aligned}
& \frac{1}{2\pi} \sum_{n=0,1,2,\dots} \frac{(-1)^n \sin(n\pi r)}{n} \times \\
& \frac{\omega\beta}{\omega^2 + \beta^2} \left[ -\cos(\omega t) + \cos(\omega t + \phi) - e^{-\beta t} (\cos(\phi) - 1) \right. \\
& \left. + \frac{\omega}{\beta} \left\{ -\sin(\omega t) + \sin(\omega t + \phi) - e^{-\beta t} \sin(\phi) \right\} \right] \\
& + \frac{\sin(\phi)}{\pi} \sum_{n=0,1,2,\dots} \frac{(-1)^n \sin(n\pi r)}{n} \exp \left[ -\frac{n^2 \pi^2}{\alpha} t \right] \quad (29)
\end{aligned}$$

where  $\beta = \frac{n^2 \pi^2}{\alpha}$ .