An example using Fourier series (defined in Hilbert space) to model a *stationary signal*:

$$f(x) = \begin{cases} 2.5 + 3e^{-2x}\sin 9x & 0 \le x \le \pi \\ -2.5 - 3e^{-2\pi-2x}\sin 9x & -\pi \le x \le 0 \end{cases} \tag{1}$$

Use the Fourier expansion:

$$f(x) = a_0 + \sum_n^\infty a_n \cos nx + \sum_n^\infty b_n \sin nx \tag{2}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \tag{4}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \tag{5}$$

Applying Eq(3), one finds:

$$a_0 = \frac{1}{\pi}\int_0^\pi 2.5dx + \frac{3}{\pi}\int_0^\pi e^{-2x}\sin 9x dx - \frac{1}{\pi}\int_{-\pi}^0 2.5dx - \frac{3}{\pi}e^{-2\pi}\int_{-\pi}^0 e^{-2x}\sin 9x dx \tag{6}$$

now

$$\begin{aligned}
\int_0^\pi e^{-2x}\sin 9x dx &= -\frac{1}{2}e^{-2\pi}\sin 9x\Big|_0^\pi + \frac{9}{2}\int_0^\pi e^{-2x}\cos 9x dx && (7)\\
&= -\frac{9}{4}e^{-2x}\cos 9x\Big|_0^\pi - \frac{81}{4}\int_0^\pi e^{-2x}\sin 9x dx && (8)\\
&= -\frac{9}{85}e^{-2x}\cos 9x\Big|_0^\pi && (9)\\
&= \frac{9}{85}\left[e^{-2\pi}+1\right] && (10)\\
&= -e^{-2\pi}\int_{-\pi}^0 e^{-2x}\sin 9x dx && (11)\\
& && (12)
\end{aligned}$$

and since the first and third integrals of Eq(6) cancel, one finds:

$$a_0 = \frac{54}{85\pi}\left[e^{-2\pi}+1\right] \tag{13}$$

Now $a_n =$

$$\frac{1}{\pi}\int_0^\pi 2.5\cos nx dx + \frac{3}{\pi}\int_0^\pi e^{-2x}\sin 9x\cos nx dx$$
$$-\frac{1}{\pi}\int_{-\pi}^0 2.5\cos nx dx - \frac{3}{\pi}e^{-2\pi}\int_{-\pi}^0 e^{-2x}\sin 9x\cos nx dx \tag{14}$$

Using the identity,

$$\cos nx \sin \beta x = \frac{1}{2}\sin(\beta + n)x + \frac{1}{2}\sin(\beta - n)x \tag{15}$$

we may write:

$$\frac{3}{\pi}\int_0^{\pi} e^{-2x}\sin 9x \cos nx dx = \frac{3}{2\pi}\int_0^{\pi} e^{-2x}\sin(9+n)x dx + \frac{3}{2\pi}\int_0^{\pi} e^{-2x}\sin(9-n)x dx \tag{16}$$

Let $9 \pm n = \alpha$;

$$\frac{3}{2\pi}\int_0^{\pi} e^{-2x}\sin \alpha x dx \tag{17}$$

$$= -\frac{3}{4\pi}e^{-2x}\sin \alpha x\Big|_0^{\pi} \to 0 + \frac{3\alpha}{4\pi}\int_0^{\pi} e^{-2x}\cos \alpha x dx \tag{18}$$

$$= -\frac{3\alpha}{8\pi}e^{-2x}\cos \alpha x\Big|_0^{\pi} - \frac{3\alpha^2}{8\pi}\int_0^{\pi} e^{-2x}\sin \alpha x dx \tag{19}$$

$$= -\frac{3\alpha}{2\pi(4+\alpha^2)}e^{-2x}\cos \alpha x|_0^{\pi} \tag{20}$$

$$\tag{21}$$

and since the first and third integrals of Eq(14) cancel, we may write:

$$a_n =$$

$$-\frac{3(9+n)}{2\pi(4+(9+n)^2)}\left(e^{-2x}\cos(9+n)x|_0^{\pi}\right) - \frac{3(9-n)}{2\pi(4+(9-n)^2)}\left(e^{-2x}\cos(9-n)x|_0^{\pi}\right)$$

$$+e^{-2\pi}\frac{3(9+n)}{2\pi(4+(9+n)^2)}\left(e^{-2x}\cos(9+n)x|_{-\pi}^{0}\right) + e^{-2\pi}\frac{3(9-n)}{2\pi(4+(9-n)^2)}\left(e^{-2x} \tag{22}$$

There are two cases to consider; when n is odd, one finds that:

$$\cos(9 \pm n)x|_0^{\pi} \text{ and } \cos(9 \pm n)x|_{-\pi}^{0} = 0 \tag{23}$$

When n is even, one finds:

$$e^{-2x}\cos(9 \pm n)x\Big|_0^{\pi} = -[1 + e^{-2\pi}] = -e^{-2\pi}\left(e^{-2x}\cos(9 \pm n)x\Big|_{-\pi}^{0}\right) \tag{24}$$

Therefore,

$$a_n(n = \text{even}) =$$

$$\left(\frac{3(9+n)}{\pi(4+(9+n)^2)} + \frac{3(9-n)}{\pi(4+(9-n)^2)}\right)\left[e^{-2\pi} + 1\right] \tag{25}$$

Now $b_n =$

$$\frac{1}{\pi}\int_0^{\pi} 2.5 \sin nx dx + \frac{3}{\pi}\int_0^{\pi} e^{-2x}\sin 9x \sin nx dx$$

$$-\frac{1}{\pi}\int_{-\pi}^{0} 2.5\sin nx dx - \frac{3}{\pi}e^{-2\pi}\int_{-\pi}^{0} e^{-2x}\sin 9x \sin nx dx \tag{26}$$

For the first and third integrals (which are non-zero for n=odd) one finds:

$$\frac{-2.5}{n\pi}\cos nx\Big|_0^{\pi} + \frac{2.5}{n\pi}\cos nx\Big|_{-\pi}^{0} = \frac{10}{n\pi} \tag{27}$$

Using the identity,

$$\sin nx \sin \beta x = \frac{1}{2}\cos(\beta - n)x - \frac{1}{2}\cos(\beta + n)x \tag{28}$$

we may write:

$$\frac{3}{\pi}\int_0^{\pi} e^{-2x}\sin 9x \sin nx dx = \frac{3}{2\pi}\int_0^{\pi} e^{-2x}\cos(9-n)x dx - \frac{3}{2\pi}\int_0^{\pi} e^{-2x}\cos(9+n)x dx \tag{29}$$

Let $9 \pm n = \alpha$;

$$\frac{3}{2\pi}\int_0^{\pi} e^{-2x}\cos \alpha x dx \tag{30}$$

$$= -\frac{3}{4\pi}e^{-2x}\cos\alpha x\Big|_0^{\pi} - \frac{3\alpha}{4\pi}\int_0^{\pi} e^{-2x}\sin\alpha x dx \tag{31}$$

$$= -\frac{3}{4\pi}e^{-2x}\cos\alpha x\Big|_0^{\pi} + \frac{3}{8\pi}e^{-2x}\sin\alpha x\Big|_0^{\pi} \to 0 \tag{32}$$

$$-\frac{\alpha^2}{8\pi}\int_0^{\pi} e^{-2x}\cos\alpha x dx \tag{33}$$

$$= \frac{6}{2\pi(4+\alpha^2)}\left(e^{-2x}\cos\alpha x\big|_0^{\pi}\right) \tag{34}$$

Based on previous on the previous result for $e^{-2x}\cos\alpha x|_0^{\pi}$ one finds:

$$b_n(n = \text{ even }) =$$

$$\left(\frac{6}{\pi(4+(9+n)^2)} + \frac{6}{\pi(4+(9-n)^2)}\right)\left[e^{-2\pi}+1\right] \tag{35}$$

and thus the fourier expansion for $f(x)$ is:

$$\frac{54}{85\pi}\left[e^{-2\pi}+1\right] + \sum_{n=2,4,6..}^{\infty}\left(\frac{3(9+n)}{\pi(4+(9+n)^2)} + \frac{3(9-n)}{\pi(4+(9-n)^2)}\right)\left[e^{-2\pi}+1\right]\cos nx$$

$$\sum_{n=1,3,5..}^{\infty}\frac{10}{n\pi}\sin nx + \sum_{n=2,4,6..}^{\infty}\left(\frac{6}{\pi(4+(9+n)^2)} + \frac{6}{\pi(4+(9-n)^2)}\right)\left[e^{-2\pi}+1\right]\sin nx \tag{36}$$

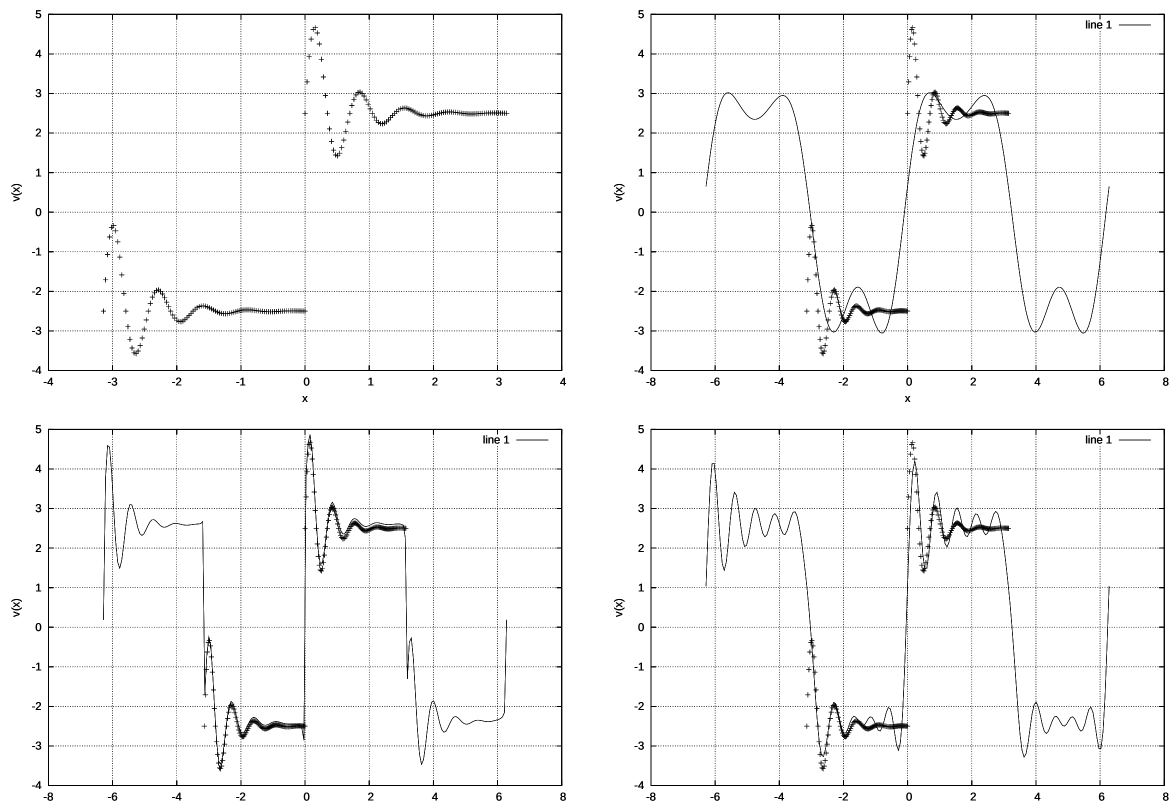

Figure 1: (Clockwise from top left): Signal due to ringdown after 74AC240 line buffer, Fourier series after 5 terms, 10 terms and 100 terms